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ON THE CANTOR–BENDIXSON DERIVATIVE, RESOLVABLE RANKS, AND PERFECT SET THEOREMS OF A. H. STONE

BY

J. CHABER AND R. POL*

Department of Mathematics, Warsaw University
ul. Banacha 2, 02-097 Warsaw, Poland
e-mail: chaber@mimuw.edu.pl e-mail: pol@mimuw.edu.pl

ABSTRACT

A Borel derivative on the hyperspace 2^X of a compactum X is a Borel monotone map $D : 2^X \to 2^X$. The derivative determines a Cantor–Bendixson type rank $\delta : 2^X \to \omega_1 \cup \{\infty\}$. We show that if $A \subset 2^X$ is analytic and $Z \subset A$ intersects stationary many layers $\delta^{-1}(\{\xi\})$, then for almost all ξ, $A \cap \delta^{-1}(\{\xi\})$ cannot be separated from $Z \cap \bigcup_{\alpha<\xi} \delta^{-1}(\{\alpha\})$ (and also from $Z \cap \bigcup_{\alpha>\xi} \delta^{-1}(\{\alpha\})$) by any F_σ-set. Another main result involves a natural partial order on 2^X related to the derivative. The results are obtained in a general framework of "resolvable ranks" introduced in the paper.

1. Introduction

Let 2^X be the hyperspace of a compact metric space X, i.e., the space of compact subsets of X with the Hausdorff metric, cf. [Ku66, §21, VII].

A Borel derivative on 2^X is a Borel map D: $2^X \to 2^X$ which is monotone, i.e., $D(K) \subset K$ for $K \in 2^X$.

An important example of a Borel derivative is the Cantor-Bendixson derivative $D(K) = K'$, where K' is the set of accumulation points of K, cf. [Ku68, §43,

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VIII, 2]. An illuminating presentation of Borel derivatives is given by Kechris [Ke94, 34].

For a Borel derivative D: $2^X \to 2^X$ the αth iterated derivative D^α: $2^X \to 2^X$ is defined inductively as follows: $D^0(K) = K$, $D^{\alpha+1}(K) = D(D^\alpha(K))$ and $D^\alpha(K) = \bigcap_{\beta<\alpha} D^\beta(K)$, for limit α. Each D^α is a Borel map [Ku68, §43, I], cf. also [CM83], where the Borel complexity of the iterations is investigated.

The rank $\delta : 2^X \to \omega_1 \cup \{\infty\}$ determined by a Borel derivative D on 2^X assigns to a compact set K in X the minimal ξ with $D^{\xi+1}(K) = \emptyset$, if such a ξ exists, or ∞, if $D^\xi(K) \neq \emptyset$ for all ξ.

The following theorem is one of the main results of this paper (the terminology is explained in Section 2):

1.1 THEOREM: *Let δ: $2^X \to \omega_1 \cup \{\infty\}$ be the rank determined by a Borel derivative D: $2^X \to 2^X$. Let A be an analytic set in 2^X and let $Z \subset A$ intersect stationary many layers $\delta^{-1}(\{\xi\})$. Then, for all but non-stationary many $\xi \in \omega_1$, each F_σ-set containing $A \cap \delta^{-1}(\{\xi\})$ intersects both sets $Z \cap \bigcup_{\alpha<\xi} \delta^{-1}(\{\alpha\})$ and $Z \cap \bigcup_{\alpha>\xi} \delta^{-1}(\{\alpha\})$.*

For each ξ satisfying the assertion of Theorem 1.1, the analytic set $A \cap \delta^{-1}(\{\xi\})$ is uncountable, hence it contains a Cantor set. Some stronger conclusions along this line are established in Theorem 1.2 below.

A result analogous to Theorem 1.1 (with a somewhat weaker assertion) was obtained in a joint paper by G. Gruenhage and the authors [ChGP95, Corollary 1.3] for the rank δ: $WO \to \omega_1$, where WO is the space of well-ordered subsets of the rationals $\mathbf{Q}$ and $\delta(A)$ is the order type of A in WO. In this case the layer $\delta^{-1}(\{\xi\})$ is the ξth Lusin's constituent WO_ξ, cf. [Ku66, §3, XV], [Ke94, 27.C,D].

The set WO has a natural partial order $\preceq$, see [Ku66, §30, XII (1)], and it was proved in [ChGP98, Proposition 5.3] that any Souslin set in WO intersecting stationary many constituents WO_ξ contains an $\preceq$-antichain intersecting all but non-stationary many constituents in a Cantor set.

The next theorem provides a counterpart to this result for the Borel derivatives, where we associate with a Borel derivative D: $2^X \to 2^X$ a partial order $\preceq$ on 2^X defined by

$$L \preceq K \quad \text{iff } L = D^\alpha(K) \text{ for some } \alpha.$$

1.2 THEOREM: *Let D: $2^X \to 2^X$ be a Borel derivative and let $\preceq$ and δ be the partial order and the rank determined by D. Then each analytic set $A \subset 2^X$ intersecting stationary many layers $\delta^{-1}(\{\xi\})$ contains an $\preceq$-antichain intersecting all but non-stationary many layers $\delta^{-1}(\{\xi\})$ in a Cantor set.*

Our proofs will be based on the fact (proved in Section 3) that for any Borel derivative $D\colon 2^X \to 2^X$ there is a continuous map $\pi\colon M \to 2^X$ from a complete metric space of weight $\aleph_1$ such that the layers $(\delta \circ \pi)^{-1}(\{\xi\})$ form a "natural stratification" of M; cf. Section 2. This allows one to use non-separable Borel theory in M, notably some variations of A. H. Stone perfect set theorems, to get in effect Theorems 1.1 and 1.2.

Such an approach was originated in [ChGP95, ChGP98] to obtain the above-mentioned results about Lusin's constituents.

It is useful to carry out the arguments involving non-separable Borel theory in a more general setting, considering ranks $\delta\colon E \to \omega_1$ which admit continuous parametrizations $\pi\colon M \to E$ on completely metrizable spaces of weight $\aleph_1$ with the layers of $\delta \circ \pi\colon M \to \omega_1$ forming appropriate stratifications of M. We shall call such ranks resolvable.

In Section 2 we clarify the terminology and we set some background for further discussion. In Section 3 we introduce the notion of resolvable ranks and we check that the rank determined by a Borel derivative is resolvable. Then, in Sections 4 and 5, we establish several facts about resolvable ranks which imply Theorems 1.1 and 1.2, respectively. In Section 6 we prove a Hurewicz-type result hinting at some possible extensions of Theorem 1.1. The last section contains some additional information concerning the subject of this note. In particular, we provide there more examples of resolvable ranks and we generalize Theorems 1.1 and 1.2 to the class of σ-compact spaces. Some other results involving resolutions for Lusin–Sierpiński indices on coanalytic sets can be found also in [ChP].

2. Terminology and some auxiliary facts

Souslin and analytic sets. Let X be a topological space. A set $S \subset X$ is Souslin in X if there is a Borel set B in the product $X \times \mathbf{N}^{\mathbf{N}}$ by the irrationals which projects onto S.

Let X be metrizable. Then B in the definition of Souslin sets can be chosen as a G_δ-set. The sets in X which are Souslin in a completion of X (no specific completion makes a difference) are called absolutely Souslin; in this case B can be chosen completely metrizable, i.e., an absolute G_δ-set. If both $S \subset X$ and $X \setminus S$ are Souslin, S is called bi-Souslin. If X is a separable completely metrizable space, the Souslin sets in X coincide with the analytic sets in X, i.e., continuous images of the irrationals, and the bi-Souslin sets are Borel, cf. [Ku66, Ke94, Ha92].

STATIONARY SETS IN ω_1. A set of countable ordinals $\Gamma \subset \omega_1$ is stationary if Γ intersects each c.u.b. set in ω_1, i.e., each closed unbounded set, cf. [Kun 80, II, §6]. We shall say that a property $\mathcal{P}(\xi)$ holds true for all but non-stationary many ordinals ξ in ω_1 if the set of ordinals ξ for which $\mathcal{P}(\xi)$ fails is non-stationary in ω_1.

NATURAL STRATIFICATIONS OF METRIZABLE SPACES OF WEIGHT $\aleph_1$. The Baire space $B(\aleph_1)$ is the countable product of the discrete space of cardinality $\aleph_1$. The points of $B(\aleph_1)$ are functions $s\colon \mathbf{N} \to \omega_1$ and we let

$$\kappa(s) = \min\{\alpha : s(\mathbf{N}) \subset [0, \alpha)\}, \tag{1}$$

$$B_\xi = \kappa^{-1}(\{\xi\}). \tag{2}$$

Then

$$\bigcup_{\alpha \leq \xi} B_\alpha \text{ is separable and closed for } \xi < \omega_1, \tag{3}$$

and

$$\bigcup_{\alpha \leq \xi} B_\alpha = \overline{\bigcup_{\alpha < \xi} B_\alpha} \text{ for limit } \xi. \tag{4}$$

Let X be a metrizable space of weight $\aleph_1$, let $\{x_\alpha : \alpha < \omega_1\}$ be dense in X, $\lambda(x) = \min\{\xi : x \in \overline{\{x_\alpha : \alpha < \xi\}}\}$, and let

$$P_\xi = \lambda^{-1}(\{\xi\}). \tag{5}$$

Then $\{P_\xi : \xi < \omega_1\}$ is a decomposition of X satisfying (3) and (4) with B_α replaced by P_α. Each such decomposition will be called a natural stratification of X. For any two natural stratifications $\{P_\xi : \xi < \omega_1\}$ and $\{P'_\xi : \xi < \omega_1\}$ of X,

$$P_\xi = P'_\xi \text{ and } \bigcup_{\alpha<\xi} P_\alpha = \bigcup_{\alpha<\xi} P'_\alpha \text{ for all but non-stationary many } \xi; \tag{6}$$

cf. [Po78], [ChGP95, 4.3]. In particular, the statements of the form "non-stationary many layers of X have property $\mathcal{P}$" do not depend on a specific choice of a natural stratification of X. We shall rely on this fact omitting as a rule any reference to a concrete natural stratification of X.

It will be convenient to deal with a more flexible class of decompositions of X.

2.1 Definition: A decomposition $\{P'_\xi : \xi < \omega_1\}$ of X is an **acceptable stratification** of X if it satisfies (6) for some (and hence, for an arbitrary) natural stratification $\{P_\xi : \xi < \omega_1\}$ of X.

Before establishing yet another useful fact, let us recall that if Y is a selector for a natural stratification, then σ-discrete subsets of Y are exactly the ones intersecting only non-stationary many layers; cf. [Po78, Sec. 2.3].

2.2 LEMMA: *Assume that $f\colon X' \to X$ is a continuous mapping between metrizable spaces, $f^{-1}(x)$ is separable for $x \in X$, and $f(F)$ is σ-discrete for any σ-discrete set $F \subset X'$. If $\{P_\xi : \xi < \omega_1\}$ is a natural stratification of X then $\{P'_\xi : \xi < \omega_1\}$, where $P'_\xi = f^{-1}(P_\xi)$ for $\xi < \omega_1$, is an acceptable stratification of X'.*

Proof: Observe that $\{P'_\xi : \xi < \omega_1\}$ satisfies (3) with B_α replaced by P'_α. Hence the layers (5) of λ defined by $\lambda(x) = \min\{\xi : x \in \overline{\bigcup_{\alpha<\xi} P'_\alpha}\}$ form a natural stratification of X'.

It suffices to show that $\bigcup_{\alpha\le\xi} P'_\alpha = \overline{\bigcup_{\alpha<\xi} P'_\alpha}$ for all but non-stationary many ξ; cf. (4). Indeed, in this case, the layers of λ witness that $\{P'_\xi : \xi < \omega_1\}$ is acceptable.

Suppose that for ξ in a stationary set Θ in ω_1 the sets $\bigcup_{\alpha\le\xi} P'_\alpha \setminus \overline{\bigcup_{\alpha<\xi} P'_\alpha} = \lambda^{-1}(\{\xi+1\})$ are nonempty, and let Y_Θ be a selector for the collection of these sets. Then, by the property of natural stratifications we have just recalled, Y_Θ is σ-discrete while $f(Y_\Theta)$ is not, being a selector for the family $\{P_\xi : \xi \in \Theta\}$. This contradicts the assumption that f preserves σ-discreteness. ∎

2.3 Remark: Clearly, the assertion of Lemma 2.2 still holds true if $\{P_\xi : \xi < \omega_1\}$ is only an acceptable stratification of X. In particular, if K is a copy of $B(\aleph_1)$ in a metrizable space X and $\{P_\xi : \xi < \omega_1\}$ is any acceptable stratification of X, then $\{K \cap P_\xi : \xi < \omega_1\}$ is an acceptable stratification of K.

PERFECT SET THEOREMS OF A. H. STONE. In his work concerning the non-separable Borel theory [St72], A. H. Stone proved several theorems about closed embeddings of $B(\aleph_1)$ into Souslin sets. The following result, based on Stone's ideas, is Theorem 4.1 in [ChGP95].

2.4 LEMMA: *Let S be a Souslin set in a completely metrizable space X and let $Y \subset S$. If Y is not the union of countably many locally separable sets, then S contains topologically a copy K of $B(\aleph_1)$ closed in X with $K \cap Y$ dense in K.*

The case $Y = S$ is Stone's result from [St72, 3.4]. One can relax the assumption $Y \subset S$, getting in effect a Hurewicz-type result. This is done in Proposition 6.1.

2.5 Remark: If Y is a subset of a metrizable space X of weight $\aleph_1$, then Y is the union of countably many locally separable sets if and only if Y intersects only non-stationary many layers of some (and hence, of any) natural stratification of X, cf. [Po78, Sec. 2.2].

A REMARK ON CONTINUOUS MAPPINGS OF $B(\aleph_1)$ INTO SEPARABLE METRIZABLE SPACES. The following observation made by D. Burke and the second author in [BP] will be useful in Section 4 (cf. (2) for the definition of B_ξ).

2.6 LEMMA: *Let $f\colon B(\aleph_1) \to E$ be a continuous map into a separable metrizable space and let $\mathcal{I}_\xi$ be a countable collection of closed sets in E with $\{f^{-1}(F) : F \in \mathcal{I}_\xi\}$ covering the layer B_ξ, for $\xi < \omega_1$. Then for all but non-stationary many ξ, there is an $F \in \mathcal{I}_\xi$ with the interior of $f^{-1}(F)$ in $B(\aleph_1)$ intersecting B_ξ.*

AN INDEPENDENT CANTOR SET LEMMA FOR CLOSED RELATIONS IN $B(\aleph_1)$. Let $R \subset X \times X$ be a symmetric relation in a space X. A set $C \subset X$ is R-independent if $(x, y) \notin R$ for any distinct x, y in C.

The following lemma, a consequence of Lemmas 2.6 and 2.8 in [ChGP98], is a non-separable variation of an independent Cantor set theorem due to Mycielski [My64].

2.7 LEMMA: *Let R be a closed symmetric relation in $B(\aleph_1)$. Then either there exists an open set $W \subset B(\aleph_1)$ with $W \times W \subset R$, or else all but non-stationary many layers B_ξ contain R-independent Cantor sets.*

3. Resolvable ranks

A function $\delta\colon E \to \omega_1$ will be called a rank on E; cf. [Ke94, p. 267]. We shall consider ranks which give rise to acceptable stratifications of metrizable spaces of weight $\aleph_1$; cf. Definition 2.1.

3.1 Definition: A rank $\delta\colon E \to \omega_1$ on a Hausdorff space E is **resolvable** if there exists a continuous surjection $\pi\colon M \to E$ with completely metrizable domain such that the layers $(\delta \circ \pi)^{-1}(\{\xi\})$ form an acceptable stratification of M. We shall call such π a resolution for δ.

It was noticed in [ChGP95, Sec. 2] that the rank $\delta\colon WO \to \omega_1$ which assigns the order type to well-ordered subsets of the rationals is resolvable. More generally, the Lusin–Sierpiński indices associated with Borel sieves are resolvable; cf. [ChGP95, 6.1]. Also, the Moschovakis' scales determine resolvable ranks, provided that some modest regularity conditions are met; cf. Comment 7.2.

In this section we shall show that the ranks determined by Borel derivatives are resolvable. This can be demonstrated by combining the approach of Kechris and Louveau [KL89, proof of Theorem 4] (cf. also [Ke94, 34.E]) with some arguments in [ChGP95, 6.1]. We shall present, however, a more direct method giving the resolutions which allow a transfer of the partial order determined by Borel derivatives to a Souslin relation on the domain. This additional feature will be important in our proof of Theorem 1.2.

We begin with an observation concerning the resolutions.

3.2 Remark: Any map $\pi\colon M \to E$ as in Definition 3.1, but with completeness of M weakened to the assumption that M is absolutely Souslin, gives rise to a resolution for the rank $\delta\colon E \to \omega_1$.

Indeed, let $M' \subset M \times \mathbf{N}^{\mathbf{N}}$ be an absolute G_δ-set projecting onto M, and let $\psi\colon M' \to M$ be the projection; cf. Section 2. Since ψ has separable fibers and it takes σ-discrete sets to σ-discrete sets, one can use Lemma 2.2 to conclude that $\pi \circ \psi\colon M' \to E$ is a resolution for δ.

For the rest of this section we shall fix a Borel derivative $D\colon 2^X \to 2^X$ on the hyperspace of a compact metric space.

Adopting the notation introduced in Sections 1 and 2 let us define $\Phi\colon 2^X \times B(\aleph_1) \to 2^X$ by

$$(1) \qquad \Phi(K,s) = D^{\kappa(s)}(K), \quad K \in 2^X,\ s \in B(\aleph_1).$$

We shall check that for any $\mathcal{B}$ Borel in 2^X,

$$(2) \qquad \Phi^{-1}(\mathcal{B}) \text{ is Souslin in } 2^X \times B(\aleph_1).$$

Since Borel sets in 2^X form a σ-algebra generated by the sets $\langle U\rangle = \{K \in 2^X : K \subset U\}$, with $U \subset X$ open in X, it suffices to show that for any U open in X,

$$(3) \qquad \Phi^{-1}(\langle U\rangle) \text{ is bi-Souslin in } 2^X \times B(\aleph_1).$$

Let

$$(4) \qquad V(n,\alpha) = \{s \in B(\aleph_1) : s(n) = \alpha\}.$$

For any $(K,s) \in 2^X \times B(\aleph_1)$, $\Phi(K,s) = \bigcap_n D^{s(n)+1}(K)$ and the collection $\{D^{s(n)+1}(K) : n \in \mathbf{N}\}$ is well-ordered by the inclusion. Hence

$$\Phi^{-1}(\langle U\rangle) = \{(K,s) : \textstyle\bigcap_n D^{s(n)+1}(K) \subset U\} = \bigcup_n \{(K,s) : D^{s(n)+1}(K) \subset U\}$$
$$= \textstyle\bigcup_n \bigcup_{\alpha<\omega_1} (D^{\alpha+1})^{-1}(\langle U\rangle) \times V(n,\alpha).$$

For each fixed n, the collection $\{V(n,\alpha) : \alpha < \omega_1\}$ is discrete in $B(\aleph_1)$ and each map $D^\alpha : 2^X \to 2^X$ is Borel. Therefore $\Phi^{-1}(\langle U \rangle)$ is a countable union of sets, each being a discrete union of Borel sets. This demonstrates (3) and completes the proof of (2).

Consider

$$\mathcal{M} = \{(K,s) : \delta(K) = \kappa(s)\}. \tag{5}$$

Then

$$\mathcal{M} \text{ is Souslin in } 2^X \times B(\aleph_1). \tag{6}$$

To see this, let us notice that $\mathcal{D} = \{L \in 2^X : L \neq \emptyset,\ D(L) = \emptyset\}$ is Borel in 2^X, and, cf. (1), $\mathcal{M} = \Phi^{-1}(\mathcal{D})$. Therefore, $\mathcal{M}$ is Souslin, by (2).

Having established (6), we can now show that for

$$\mathcal{E} = \{K \in 2^X : 0 < \delta(K) < \infty\}, \tag{7}$$

the rank

$$\delta\colon \mathcal{E} \to \omega_1 \text{ is resolvable;} \tag{8}$$

cf. Comment 7.6.

Let $\pi\colon \mathcal{M} \to \mathcal{E}$ and $\psi\colon \mathcal{M} \to B(\aleph_1)$ be the projections onto the first and the second coordinate, respectively. Then, by (5), since $\kappa(s) > 0$ for $s \in B(\aleph_1)$,

$$\pi(\mathcal{M}) = \delta^{-1}(\{\xi : 0 < \xi < \omega_1\}) = \mathcal{E}, \tag{9}$$

and

$$(\delta \circ \pi)^{-1}(\{\xi\}) = \psi^{-1}(B_\xi), \qquad \text{for } \xi > 0, \tag{10}$$

where B_ξ are the layers of $B(\aleph_1)$ defined in Section 2(2).

Remark 3.2 combined with (6), (9), (10), and Lemma 2.2 applied to ψ, demonstrate (8).

We shall close this section with a verification that for the order $\preceq$ determined by the derivative D, and the projection $\pi\colon \mathcal{M} \to \mathcal{E}$,

$$\mathcal{G} = \{(u,v) : \pi(u) \preceq \pi(v)\} \text{ is Souslin in } \mathcal{M} \times \mathcal{M}. \tag{11}$$

To this end, let us notice that the set $\mathcal{G}_\beta = \{(L,K) : L = D^\beta(K)\}$, i.e., the reflected graph of the βth iterate of D, is Borel, cf. Section 1, and that for any $(L,t),(K,s) \in \mathcal{M}$ with $L = D^\beta(K)$ we have $D(L) \neq \emptyset$, as $\delta(L) > 0$, and hence $\beta < \delta(K) = \kappa(s)$.

Therefore, with $V(n,\alpha)$ defined in (4), one can write

$$\begin{aligned}\mathcal{G} &= \{\langle (L,t),(K,s)\rangle \in \mathcal{M}\times\mathcal{M} : L = D^{\beta}(K),\ \beta \le s(n) \text{ for some } n\in \mathbf{N}\}\\ &= \mathcal{M}\times\mathcal{M}\cap \textstyle\bigcup_n \bigcup_\alpha (\bigcup_{\beta\le\alpha} \mathcal{G}_\beta \times B(\aleph_1)\times V(n,\alpha)),\end{aligned}$$

where $\langle (L,t),(K,s)\rangle$ is identified with $\langle (L,K),t,s\rangle$.

Since each collection $\{V(n,\alpha) : \alpha < \omega_1\}$ is discrete in $B(\aleph_1)$, $\mathcal{G}$ is a σ-discrete union of Borel sets in $\mathcal{M}\times\mathcal{M}$, which proves (11).

4. Proof of Theorem 1.1

We shall present a proof in a more general framework of resolvable ranks. Since, as was established in Section 3, the ranks determined by Borel derivatives are resolvable, Proposition 4.1 implies readily Theorem 1.1, in fact, with a slightly stronger assertion.

4.1 Proposition: *Let δ: $E \to \omega_1$ be a resolvable rank on a separable metrizable space E, let A be a Souslin set in E and let $Z \subset A$ intersect stationary many layers $\delta^{-1}(\{\xi\})$. Then, for all but non-stationary many $\xi \in \omega_1$, if $A\cap\delta^{-1}(\{\xi\})$ is covered by sets $F_0, F_1, \dots$ closed in E, then some F_n intersects simultaneously $Z\cap\bigcup_{\alpha<\xi}\delta^{-1}(\{\alpha\})$ and $Z\cap\bigcup_{\alpha>\xi}\delta^{-1}(\{\alpha\})$.*

Proof: Let π: $M \to E$ be a resolution for the rank δ and let

$$Y = \pi^{-1}(Z). \tag{1}$$

Using Remark 2.5 and Lemma 2.4 with $S = \pi^{-1}(A)$ one gets a copy K of $B(\aleph_1)$ closed in M, such that

$$K = \overline{K\cap Y} \subset \pi^{-1}(A). \tag{2}$$

Let $Y_\xi = K\cap Y\cap\bigcup_{\alpha<\xi}(\delta\circ\pi)^{-1}(\{\alpha\})$, $\lambda(x) = \min\{\xi : x\in\overline{Y_\xi}\}$ for $x\in K$, and let us consider the natural stratification of K defined by λ, as in Section 2(5). By Remark 2.3 (cf. also (4) in Section 2) there exists a c.u.b. set Γ such that, for any relatively open V in K,

$$V\cap(\delta\circ\pi)^{-1}(\{\xi\})\neq\emptyset \text{ implies } V\cap Y\cap\bigcup_{\alpha<\xi}(\delta\circ\pi)^{-1}(\{\alpha\})\neq\emptyset \text{ for } \xi\in\Gamma. \tag{3}$$

Having set a background, let us assume, aiming at a contradiction, that for each ξ in a stationary set Θ in ω_1 there exists a countable collection $\mathcal{I}_\xi$ of closed sets in E such that $A\cap\delta^{-1}(\{\xi\})\subset\bigcup\mathcal{I}_\xi$ and, for each $F\in\mathcal{I}_\xi$, either

$$F\cap Z\cap\bigcup_{\alpha<\xi}\delta^{-1}(\{\alpha\}) = \emptyset \quad\text{or}\quad F\cap Z\cap\bigcup_{\alpha>\xi}\delta^{-1}(\{\alpha\}) = \emptyset. \tag{4}$$

Then, Lemma 2.6 applied to the restriction $\pi|K : K \to E$, the inclusion in (2), and Remark 2.3 provide $\xi \in \Theta \cap \Gamma$, $F \in \mathcal{I}_\xi$ and a relatively open set V in K with $V \cap (\delta \circ \pi)^{-1}(\{\xi\}) \neq \emptyset$ and $\pi(V) \subset F$.

By (1) and (3),

$$F \cap Z \cap \bigcup_{\alpha<\xi} \delta^{-1}(\{\alpha\}) \neq \emptyset, \tag{5}$$

and since V is non-separable, (2) implies that $V \cap Y \setminus \bigcup_{\alpha\leq\xi}(\delta \circ \pi)^{-1}(\{\alpha\}) \neq \emptyset$, hence, again by (1), also

$$F \cap Z \cap \bigcup_{\alpha>\xi} \delta^{-1}(\{\alpha\}) \neq \emptyset. \tag{6}$$

However, (5) and (6) contradict (4), which completes the proof. ∎

5. Proof of Theorem 1.2

Souslin quasi-orders on $B(\aleph_1)$ respecting the layer structure. We say that a relation $\preceq$ is a quasi-order, cf. [KM76, II, §9], if it is reflexive and transitive, but not necessarily strict, i.e., we allow both $x \preceq y$ and $y \preceq x$ for distinct x, y. A set C is a chain (an antichain) with respect to $\preceq$ if for any (for no) distinct x, y in C, $x \preceq y$ or $y \preceq x$. We say that x, y have a common extension, if there is a z with $x \preceq z$ and $y \preceq z$.

Let $\preceq$ be a quasi-order on $B(\aleph_1)$. We shall say that $\preceq$ is Souslin if

$$R = \{(s,t) : s \preceq t \text{ or } t \preceq s\} \text{ is Souslin in } B(\aleph_1) \times B(\aleph_1), \tag{1}$$

and we shall call $\preceq$ proper if

$$R_\Delta = \{(s,t) : s \preceq t \text{ and } t \preceq s\} \text{ is closed in } B(\aleph_1) \times B(\aleph_1). \tag{2}$$

Recall that B_ξ are the layers of $B(\aleph_1)$ defined in Section 2(2). We shall say that $\preceq$ respects the layer structure in $B(\aleph_1)$ if there exists a c.u.b. set Γ in ω_1 such that

$$\text{if } \{s,t\} \subset B_\xi \text{ is an antichain and } \xi \in \Gamma, \text{ then } s,t \text{ have no common extension,} \tag{3}$$

$$\text{if } s \in B_\xi \text{ and } \xi \in \Gamma, \text{ then } \{t : t \preceq s\} \subset \bigcup_{\alpha\leq\xi} B_\alpha; \tag{4}$$

cf. [ChGP98, Sec. 5].

5.1 Remark: Let K be a copy of $B(\aleph_1)$ in $B(\aleph_1)$. If $\preceq$ is a quasi-order on $B(\aleph_1)$ satisfying (1)–(4), then the restriction of $\preceq$ to K also satisfies (1)–(4), as by Remark 2.3 properties (3) and (4) do not depend on any specific choice of the homeomorphism of $B(\aleph_1)$ onto K.

5.2 LEMMA: *Let $\preceq$ be a proper Souslin quasi-order on $B(\aleph_1)$, respecting the layer structure in $B(\aleph_1)$. Then either there is a stationary set of layers, each containing a Cantor set which is an R_Δ-independent $\preceq$-chain, or there is an $\preceq$-antichain intersecting all but non-stationary many layers in a Cantor set.*

Proof: The main idea of the proof is similar to that in the proof of Lemma 5.2 in [ChGP98]. Assume that the first part of the alternative fails. We shall construct an antichain satisfying the conditions of the second part.

Let $\mathcal{K}$ denote the collection of closed subsets of $B(\aleph_1)$ homeomorphic to $B(\aleph_1)$ and let $K \in \mathcal{K}$. We say that $\preceq$ splits K if there is an uncountable discrete collection $\mathcal{K}_K \subset \mathcal{K}$ of subsets of K such that every selector for $\mathcal{K}_K$ is an $\preceq$-antichain.

Assume first that $\preceq$ splits every element of $\mathcal{K}$. This leads immediately to a sequence $\mathcal{K}_0, \mathcal{K}_1, \ldots$ of discrete subcollections of $\mathcal{K}$ such that each $K \in \mathcal{K}_{n-1}$ contains uncountably many elements of $\mathcal{K}_n$, $\mathcal{K}_n$ refines $\mathcal{K}_{n-1}$, each selector for $\mathcal{K}_n$ is an antichain, and $\operatorname{diam}(K) < 1/n$ for K in $\mathcal{K}_n$. Then $L = \bigcap_n \bigcup \mathcal{K}_n$ is a closed copy of $B(\aleph_1)$ which is an $\preceq$-antichain. By Remark 2.3, $L \cap B_\xi$ contains a Cantor set for all but non-stationary many ξ. Thus the proof is completed in this case.

Suppose now that $\preceq$ does not split some $K \in \mathcal{K}$. To simplify the notation, assume that $K = B(\aleph_1)$; cf. Remark 2.3.

Observe that, by (4), every open $W \subset B(\aleph_1)$ with $W \times W \subset R_\Delta$, cf. (2), is separable. Thus, by Lemma 2.7, for all but non-stationary many ξ, there exists an R_Δ-independent Cantor set $C_\xi \subset B_\xi$. Furthermore, a theorem of Galvin, cf. [Ke94, 19.7], applied to R restricted to C_ξ yields, by (1), a Cantor set C'_ξ which is either a $\preceq$-chain or it is an $\preceq$-antichain. Since we assumed that the first part of the alternative in Lemma 5.2 is not true, we are left with all but non-stationary many C'_ξ being the antichains. To simplify the notation we shall assume that

(5) $$C_\xi \subset B_\xi \text{ is an } \preceq\text{-antichain for } \xi \in \Gamma,$$

where Γ has also the properties described in (3) and (4).

For any compact set $C \subset B(\aleph_1)$, let us put

(6) $$R(C) = \{t : (s,t) \in R \text{ for some } s \in C\}.$$

Then $R(C)$ is Souslin, being the projection of a Souslin set $R \cap (C \times B(\aleph_1))$ parallel to the compact axis.

For each $\alpha \in \Gamma$, let

$$\mathcal{C}_\alpha = \{C \subset C_\alpha : C \text{ is a Cantor set and } R(C) \text{ intersects only non-stationary many layers } B_\xi\}. \tag{7}$$

Assume that $\mathcal{C}_\alpha \neq \emptyset$ for $\alpha \in \Gamma$. Choose $C'_\alpha \in \mathcal{C}_\alpha$ and let Λ_α be a c.u.b. set such that

$$\text{for } \xi \in \Lambda_\alpha, \quad R(C'_\alpha) \cap B_\xi = \emptyset. \tag{8}$$

Let Λ be the diagonal intersection of the sets $\Lambda_\alpha \cap \Gamma$, cf. [Kun80, II, 6.14], i.e., Λ is a c.u.b. set such that

$$\text{if } \alpha < \xi, \quad \alpha, \xi \in \Lambda, \quad \text{then } \xi \in \Lambda_\alpha. \tag{9}$$

Then $\bigcup\{C'_\xi : \xi \in \Lambda\}$ is an $\preceq$-antichain. Indeed, by (5), all C'_ξ are antichains, and (8) and (9) guarantee that if $s \in C'_\alpha$, $t \in C'_\xi$, with $\alpha < \xi$ and $\alpha, \xi \in \Lambda$, then s and t are $\preceq$-incomparable.

Thus to finish the proof, it remains to consider the situation when there exists a $\xi \in \Gamma$ with $\mathcal{C}_\xi = \emptyset$. We shall show that this leads to a contradiction with our assumption that $\preceq$ does not split $B(\aleph_1)$.

To this end, divide C_ξ into disjoint Cantor sets $\{C_r : r \in 2^{\mathbf{N}}\}$. Since $\mathcal{C}_\xi = \emptyset$, by (7) and Stone's theorem, cf. 2.4 and 2.5, there exist $K_r \in \mathcal{K}$ such that

$$K_r \subset R(C_r) \setminus \bigcup_{\alpha \leq \xi} B_\alpha, \quad \text{for } r \in 2^{\mathbf{N}}.$$

By (6), (4) and (3), every selector for the collection $\{K_r : r \in 2^{\mathbf{N}}\}$ is an $\preceq$-antichain. We shall find an uncountable discrete refinement of this collection consisting of elements of $\mathcal{K}$.

Let d be a metric on $B(\aleph_1)$. For each $r \in 2^{\mathbf{N}}$ let F_r be an uncountable $1/n_r$-discrete subset of K_r. Fix a natural n and an uncountable set $\{r_\alpha : \alpha < \omega_1\} \subset 2^{\mathbf{N}}$ such that $n_{r_\alpha} = n$ for $\alpha < \omega_1$. By induction on $\alpha < \omega_1$ we choose $s_\alpha \in F_{r_\alpha}$ such that the set $\{s_\alpha : \alpha < \omega_1\}$ is $1/2n$-discrete.

For each $\alpha < \omega_1$ let L_α be a closed-and-open neighborhood of s_α in K_{r_α} with $\operatorname{diam}(L_\alpha) < 1/6n$. Clearly, $L_\alpha \in \mathcal{K}$ and the collection $\{L_\alpha : \alpha < \omega_1\}$ is $1/6n$-discrete. Therefore $B(\aleph_1)$ is split, a contradiction ending the proof. ∎

Proof of Theorem 1.2: As in Section 4, we shall present a proof in the framework of resolvable ranks. The generality does not complicate the matter, enabling us to encompass the corresponding result for Lusin's constituents [ChGP98, Proposition 5.3]; cf. also Comment 7.1.

Let $\delta\colon E \to \omega_1$ be a resolvable rank. We shall say that a partial order $\preceq$ on E is resolvable simultaneously with the rank δ if there exists a resolution $\pi\colon M \to E$ for δ such that

$$(10) \qquad \{(u,v) \in M \times M : \pi(u) \preceq \pi(v)\} \text{ is Souslin in } M \times M.$$

If the partial order $\preceq$ and the rank δ are determined by a Borel derivative $D\colon 2^X \to 2^X$, cf. Section 1, then, by Section 3(8),(11), $\preceq$ is resolvable simultaneously with δ on $\mathcal{E}$. Moreover, the initial segments $\{D^\alpha(K) : \alpha < \delta(K)\}$ of $\preceq$ on $\mathcal{E}$ are linearly ordered by $\preceq$ and δ is non-decreasing with respect to $\preceq$. Thus Proposition 5.3 below implies Theorem 1.2.

5.3 PROPOSITION: *Let $\delta\colon E \to \omega_1$ be a resolvable rank on E and let $\preceq$ be a partial order on E resolvable simultaneously with δ. If the initial segments of $\preceq$ are countable and linearly ordered by $\preceq$, and δ is non-decreasing with respect to $\preceq$, then each Souslin set $A \subset E$ intersecting stationary many layers $\delta^{-1}(\{\xi\})$ contains an $\preceq$-antichain intersecting all but non-stationary many layers $\delta^{-1}(\{\xi\})$ in a Cantor set.*

Proof: Let $\pi\colon M \to E$ be a resolution for δ satisfying (10) and let A be a Souslin set in E intersecting stationary many layers $\delta^{-1}(\{\xi\})$. Then $S = \pi^{-1}(A)$ is a Souslin set in M and by 2.5 and 2.4 there exists an embedding $h\colon B(\aleph_1) \to S$.

Consider the quasi-order $\preceq$ on $B(\aleph_1)$ defined by

$$(11) \qquad s \preceq t \quad \text{iff } (\pi \circ h)(s) \preceq (\pi \circ h)(t).$$

By Remark 2.3 applied to $K = h(B(\aleph_1))$, $\preceq$ is a quasi-order respecting the layer structure of $B(\aleph_1)$. Condition (10) implies that $\preceq$ satisfies (1), and the continuity of $\pi \circ h$ gives (2). Thus $\preceq$ satisfies the assumptions of Lemma 5.2.

If a Cantor set C in $B(\aleph_1)$ is R_Δ-independent, where R_Δ is the relation given by (2), then $\pi \circ h$ maps C injectively onto a Cantor set $H \subset A$. Observe that, by (11), C is a $\preceq$-chain (antichain) exactly when so is H.

By (10) and (11), the set $\{(x,y) \in H \times H : x \preceq y\}$ is analytic as the continuous image of the restriction of $\preceq$ in $B(\aleph_1)$ to $C \times C$. Since the initial segments of $\preceq$ on E are countable, the relation $\preceq$ restricted to $H \times H$ is of the first category, by the Kuratowski–Ulam theorem, cf. [Ku66, §22, VI]. It follows that H is not a $\preceq$-chain and consequently, the first part of the alternative in the assertion of

Lemma 5.2 is false. We are left with the second part of the assertion, which, in view of Remark 2.3, implies the assertion of Proposition 5.3. ∎

6. Hurewicz-type results

The assertion of Proposition 4.1 yields the following Hurewicz-type result: for all but non-stationary many ξ there exists a Cantor set $C_\xi \subset (A \cap \delta^{-1}(\{\xi\})) \cup Z$ with both $C_\xi \cap \bigcup_{\alpha<\xi} \delta^{-1}(\{\alpha\})$ and $C_\xi \cap \bigcup_{\alpha>\xi} \delta^{-1}(\{\alpha\})$ countable and dense in C_ξ.

To get such Cantor sets, one can apply a variation of a Kechris–Louveau–Woodin theorem [Ke94, 21.22] described in [MP95]. Alternatively, one can also follow the idea of Solecki's proof of this theorem [So94, Corollary 8] (notice that one can replace in this proof the use of [So94, Theorem 1] with some more direct arguments). Along these lines, we shall refine now Lemma 2.4 by relaxing the assumption that $Y \subset S$, and we shall use this result to indicate some variation of Proposition 4.1 with Z not necessarily contained in A. In our proof, Stone's techniques will be combined with some ideas from [So94, Corollary 8].

Let $Q = \{s \in B(\aleph_1) :$ the support of s is finite$\}$. In the sequel we shall concentrate on Q, but, in fact, any dense σ-discrete subset of $B(\aleph_1)$ can be used to the same effect; cf. Comment 7.7.

6.1 Proposition: *Let S be a Souslin set in a completely metrizable space X and let $Y \subset X$. Then either there exists an F_σ-set F disjoint from Y such that $S \setminus F$ can be decomposed into countably many locally separable sets, or else there exists a homeomorphic embedding h of $B(\aleph_1)$ onto a closed in X subset of $S \cup Y$ with $h(Q) \subset Y$ and $h(B(\aleph_1) \setminus Q) \subset S$.*

Proof: Put $X' = X \times 2^{\mathbf{N}}$ and let $\psi\colon X' \to X$ be the projection, S' a G_δ-subspace of X' with $\psi(S') = S$, and $Y' = \psi^{-1}(Y)$.

Let $\mathcal{I} = \{W \subset X' : W \cap S' = \emptyset$ or $W \cap Y' = \emptyset\}$ and let $\mathcal{I}^*$ be the collection $\mathcal{I}$ enriched by separable subsets of X'. By a standard transfinite exhaustion procedure, cf. [St63, Theorem 4'], one can write $X' = K \cup J$, where J is the union of a σ-discrete subcollection of closed sets from $\mathcal{I}^*$, and K is a closed set with no nonempty relatively open subset in $\mathcal{I}^*$. If K is empty, then the first part of the alternative holds. Otherwise, K is nowhere locally separable,

(1) $$H = K \cap Y' \text{ is dense in } K$$

and

$$G = K \cap S' = \bigcap_{n \geq 0} G_n \text{ with } G_0 \supset G_1 \supset \cdots \text{ open and dense in } K. \tag{2}$$

Let $n \geq 0$, let U be open in K and let $x_0 \in U \cap H$. Since no nonempty open set in K is separable and ψ is perfect, there exist, for $0 < \xi < \omega_1$, $x_\xi \in U \cap G_n$ such that $\{\psi(x_\xi) : \xi < \omega_1\}$ is discrete in X as an indexed collection of singletons. Thus we can find a family $\mathcal{U} = \{U_\xi : \xi < \omega_1\}$ of open subsets of K such that $x_\xi \in U_\xi \subset \overline{U_\xi} \subset U$ for $\xi < \omega_1$, $U_\xi \subset G_n$ for $\xi > 0$, and $\psi(\mathcal{U}) = \{\psi(U_\xi) : \xi < \omega_1\}$ is discrete in X. Finally, we define $y_0 = x_0 \in U_0 \cap H$ and we use (1) to choose $y_\xi \in U_\xi \cap H$ for $\xi > 0$.

Therefore, starting with $n = 0$, $U(\emptyset) = K$ and arbitrary $y(\emptyset) \in H$ we can define, by repeating the above observation, collections $\mathcal{U}_n = \{U(\tau) : \tau \in (\omega_1)^n\}$ of open subsets of K with $\operatorname{mesh}(\mathcal{U}_{n+1}) \leq 1/(n+1)$, the closures of $\mathcal{U}_{n+1}$ in K refining $\mathcal{U}_n$, and $\psi(\mathcal{U}_n)$ discrete in X. Moreover, in the inductive construction we fix, for each τ, a point $y(\tau) \in U(\tau) \cap H$ and we make sure that

$$y(\tau^\frown 0) = y(\tau) \in U(\tau^\frown 0), \tag{3}$$

$$U(\tau^\frown \xi) \subset G_{|\tau|} \quad \text{for } \xi > 0, \tag{4}$$

where $|\tau|$ denotes the length of τ.

Let $f\colon B(\aleph_1) \to K$ be defined by the condition $f(t) \in \bigcap_{n \geq 0} \overline{U(t|n)}$. Then $L = f(B(\aleph_1)) = \bigcap_{n \geq 0} \overline{\bigcup \mathcal{U}_n}$ is a closed in K copy of $B(\aleph_1)$ and $h = \psi \circ f$ is a homeomorphic embedding of $B(\aleph_1)$ onto a closed subset of X. By (3) and (1), $f(Q) \subset H \subset Y'$, therefore $h(Q) \subset Y$. If $t \in B(\aleph_1) \setminus Q$, then (4) gives $U(t|n+1) \subset G_n$ for infinitely many n, hence by (2), $f(t) \in G \subset S'$, consequently $h(t) \in S$ and the proof is completed. ∎

If the space X in Proposition 6.1 has weight $\aleph_1$, then we get, cf. [ChGP95, Theorem 1.2],

6.2 Corollary: *Let S be a Souslin set in a completely metrizable space X of weight $\aleph_1$, let $\{P_\xi : \xi < \omega_1\}$ be a natural stratification of X, and let $Y \subset X$. Then either there is an F_σ-set disjoint from Y and containing all but non-stationary many layers $S \cap P_\xi$, or else for all but non-stationary many ξ, any F_σ-set containing $S \cap P_\xi$ intersects $Y \cap \bigcup_{\alpha < \xi} P_\alpha$.*

Proof: Suppose that the first part of the alternative fails. By Remark 2.5, the first part of the alternative in Proposition 6.1 is not true, and we are left with the second part. Let ξ be a countable limit ordinal. Observe that the layer B_ξ defined in Section 2(2) is a dense G_δ in $\bigcup_{\alpha\le\xi} B_\xi$ and $Q \cap \bigcup_{\alpha<\xi} B_\xi$ is dense in $\bigcup_{\alpha\le\xi} B_\xi$. By the Baire category theorem, each F_σ-set in $B(\aleph_1)$ containing B_ξ intersects $Q \cap \bigcup_{\alpha<\xi} B_\xi$. Since $B_\xi \subset B(\aleph_1) \setminus Q$, Remark 2.3 gives the second part of the alternative. ∎

Corollary 6.2 suggests some variations of Proposition 4.1 with Z being not necessarily a subset of A.

For example, let δ: $E \to \omega_1$ be a resolvable rank, on a separable metrizable space E, let A, Z be a pair of subsets of E with A Souslin in E. Furthermore, suppose that there exists a resolution π: $M \to E$ for δ such that, for $X = M$, $S = \pi^{-1}(A)$ and $Y = \pi^{-1}(Z)$, the first part of the alternative in Corollary 6.2 fails. Then the second part of the alternative can be used to obtain an analogue of Proposition 4.1.

It would be interesting to get results in this direction not involving explicitly the resolutions. We finish this section with a brief description of an example exhibiting some essential difficulties in applying the above approach to this end.

6.3 Example: Let WO be the space of well-ordered subsets of the rationals, and let $WO_\xi = \{A \in WO : \text{type}(A) = \xi\}$ be the ξth constituent; cf. Introduction. Let us define a rank δ on $WO \times WO$ by the formula $\delta(A,B) = \max\{\text{type}(A), \text{type}(B)\}$.

Then $\mathcal{A} = \{(A,B) \in WO \times WO : \text{type}(A) \neq \text{type}(B)\}$ is Souslin, but not Borel in $WO \times WO$; cf. [Ke94, 34.C], [Ka81, Theorem 4].

However, if σ: $B(\aleph_1) \to WO$ is the standard resolution considered in [ChGP95, Sec. 2], then π: $B(\aleph_1) \times B(\aleph_1) \to WO \times WO$ is a resolution for δ such that $\pi^{-1}(\mathcal{A})$ is an F_σ-set in $B(\aleph_1) \times B(\aleph_1)$; cf. [Po79, Lemma 1].

Using the Continuum Hypothesis one can select points $p_\xi \in WO_\xi \times WO_\xi$ in such a way that $\mathcal{A}$ cannot be separated from $\mathcal{Z} = \{p_\xi : \xi < \omega_1\}$ by any F_σ-set; cf. [Ka81, Theorem 4]. It is not clear if, for every such choice of $\mathcal{Z}$, the assertion of Proposition 4.1 is true for the pair $\mathcal{A}$ and $\mathcal{Z}$.

7. Comments

7.1 GENERALIZATIONS OF THEOREMS 1.1 AND 1.2.

Another classical rank related to the Cantor–Bendixson derivative D: $2^X \to 2^X$ is δ^*: $2^X \to \omega_1$, which associates to every $K \in 2^X$ the minimal ξ with $D^{\xi+1}(K) = D^\xi(K)$. This can be put in the following more general framework.

Let $\mathcal{B} \subset 2^X$ be a Borel set. Then a Borel derivative D: $2^X \to 2^X$ determines a rank $\delta_{\mathcal{B}}$: $\mathcal{E} \to \omega_1$, where $\mathcal{E}$ is the set of all $K \in 2^X$ such that $D^\xi(K) \in \mathcal{B}$ for some ξ, and $\delta_{\mathcal{B}}(K)$ is the minimal ξ with that property. In the case of the Cantor–Bendixson derivative, the rank δ corresponds to the collection $\mathcal{B}$ of finite sets, and the rank δ^* corresponds to the collection $\mathcal{B}$ consisting of all perfect sets and the empty set. The arguments from Section 3 can be readily applied also in this situation.

To show that $\delta_{\mathcal{B}}$: $\mathcal{E} \to \omega_1$ is a resolvable rank, adopt the notation from Section 3 and consider $\mathcal{M} = \{(K,s) \in 2^X \times B(\aleph_1) : \delta_{\mathcal{B}}(K) = \kappa(s)\}$; cf. (5). Then $\mathcal{M} \subset \Phi^{-1}(\mathcal{B}) \subset \{(K,s) : \delta_{\mathcal{B}}(K) \leq \kappa(s)\}$; cf. (1). As $\{(K,s) : \delta_{\mathcal{B}}(K) < \kappa(s)\} = \bigcup_n \bigcup_{\alpha<\omega_1} \bigcup_{\beta\leq\alpha} (D^\beta)^{-1}(\mathcal{B}) \times V(n,\alpha)$ is the union of a σ-discrete collection of Borel sets, (2) implies that $\mathcal{M}$ is Souslin in $2^X \times B(\aleph_1)$; cf. (6). Thus one can use the projection π: $\mathcal{M} \to \mathcal{E}$ and Remark 3.2 to get a resolution for $\delta_{\mathcal{B}}$; cf. Comment 7.6.

Moreover, if we assume that $D^\eta(\mathcal{B}) \subset \mathcal{B}$ for $\eta < \omega_1$, the justification of condition (11) remains valid for the set $\mathcal{M}$ determined by $\delta_{\mathcal{B}}$. It follows that the partial order $\preceq$ generated by D is resolvable simultaneously with $\delta_{\mathcal{B}}$. Since $\delta_{\mathcal{B}}$ is then non-decreasing with respect to $\preceq$, the assumptions of Proposition 5.3 are satisfied by the rank determined by the set $\mathcal{B}$.

Furthermore, the compactness of X can be relaxed to σ-compactness.

Indeed, suppose that X is a σ-compact metrizable space and let 2^X denote the set of all closed subsets of X with the Effros Borel structure; cf. [Ke94, 12.8].

Fix a function D: $2^X \to 2^X$ which is Borel and monotone. Since the Effros Borel σ-algebra in 2^X is generated by the sets $\langle U \rangle = \{K \in 2^X : K \subset U\}$, with $X \setminus U$ compact, it follows that all iterations D^ξ: $2^X \to 2^X$ are Borel and, moreover, Φ: $2^X \times B(\aleph_1) \to 2^X$ defined by formula (1) in Section 3 satisfies (2) in Section 3; cf. [Ke94, 34.11].

As before, the arguments from Section 3 show that each rank $\delta_{\mathcal{B}}$ determined by D and a Borel set $\mathcal{B} \subset 2^X$ (closed with respect to all countable iterations of D) is resolvable (and satisfies the assumptions of Proposition 5.3). Thus we can use Propositions 4.1 and 5.3 to obtain the corresponding generalizations of Theorems 1.1 and 1.2.

7.2 Semiscales which determine resolvable ranks.

Let E be a subset of a completely metrizable separable space X. Following [Ke94, p. 300] we shall call a function φ: $E \to B(\aleph_1)$ a **semiscale** provided $x_i \to x$ with $x_i \in E$ and $\varphi(x_i) \to s$, imply $x \in E$. We shall consider in the sequel semiscales satisfying in addition

$$(*)\qquad \text{if } x_i \to x \text{ with } x_i \in E \text{ and } \varphi(x_i) \to s, \text{ then } \kappa(\varphi(x)) \le \kappa(s),$$

where the function $\kappa\colon B(\aleph_1) \to \omega_1$ was defined in Section 2(1). The condition $(*)$ is weaker than the semicontinuity which distinguishes the scales among semiscales; cf. [Ke94, 36.2].

PROPOSITION: *Let $\varphi\colon E \to B(\aleph_1)$ be a semiscale with property $(*)$ and let $\delta_\varphi\colon E \to \omega_1$ be defined by $\delta_\varphi(x) = \kappa(\varphi(x))$. If the fibers $\delta_\varphi^{-1}(\{\xi\})$ are absolutely Borel, then the rank δ_φ is resolvable.*

We shall sketch a proof of this fact. Let X be the completely metrizable extension of E and let $\overline{G(\varphi)}$ be the closure of the graph of φ in $X \times B(\aleph_1)$. Then, φ being a semiscale, $\overline{G(\varphi)}$ projects onto E. Consider $M = \{(x,s) \in \overline{G(\varphi)} : \delta_\varphi(x) = \kappa(s)\}$. The condition $(*)$ implies that $\overline{G(\varphi)} \subset \{(x,s) : \delta_\varphi(x) \le \kappa(s)\}$. Hence, arguing as in Comment 7.1, one can show that M is Souslin in $X \times B(\aleph_1)$ and δ_φ is a resolvable rank.

7.3 SOME CONCRETE RANKS RELATED TO THE CANTOR–BENDIXSON DERIVATIVE.

Let $D\colon 2^X \to 2^X$ be the Cantor–Bendixson derivative and let $\delta\colon 2^X \to \omega_1 \cup \{\infty\}$ be the Cantor–Bendixson rank determined by D. Hurewicz [Hu30] defined a concrete open sieve through which the complement of $\mathcal{E} = \{K : \delta(K) \ne \infty\}$ is sifted and showed that the corresponding Lusin–Sierpiński index of $K \in \mathcal{E}$ is closely related to the Cantor–Bendixson rank $\delta(K)$. Still, the resolutions for Lusin–Sierpiński indices constructed in [ChGP95, 6.1] seem not enough to get the assertions of Theorems 1.1 and 1.2 for the Cantor–Bendixson derivative directly from Hurewicz's results.

An interesting approach to the Cantor–Bendixson derivative, due to Kechris and Louveau, is presented in [Ke94, 36.G.3]. This approach can be modified to get a scale which, by Comment 7.2, gives in effect a resolution for the Cantor–Bendixson rank. However, there are some difficulties in extending this method to the case of more general derivatives, and also more adjustments are needed to get in this way resolutions suitable for Theorem 1.2.

7.4 AN INCREASING SEQUENCE OF $F_{\sigma\delta}$ SETS WHICH DETERMINES A RESOLVABLE RANK.

Let $A = \{a(\alpha) : \alpha < \omega_1\}$ be a subset of the irrationals $\mathbf{P}$ of cardinality $\aleph_1$ enumerated without repetitions. Let $E = A^{\mathbf{N}} \subset \mathbf{P}^{\mathbf{N}}$, and let $\delta\colon E \to \omega_1$ be defined by $\delta(a(\alpha_0), a(\alpha_1), \ldots) = \min\{\alpha : \alpha_n < \alpha \text{ for all } n\}$.

The rank δ is resolvable. Indeed, the map π: $B(\aleph_1) \to E$ defined by $\pi(\alpha_0, \alpha_1, \ldots) = (a(\alpha_0), a(\alpha_1), \ldots)$ satisfies $\delta \circ \pi = \kappa$, thus π is a resolution for δ; cf. Section 2(1),(2).

The sets $A_\xi = \{a(\alpha) : \alpha < \xi\}$ are countable, hence $E_\xi = A_\xi^{\mathbf{N}}$ are $F_{\sigma\delta}$-sets in $\mathbf{P}^{\mathbf{N}}$. The increasing sequence $E_1 \subset \cdots \subset E_\xi \subset \cdots$ determines the rank δ, as $\delta(x) = \min\{\xi : x \in E_\xi\}$ for $x \in E$.

If we require in addition that A is a rarefied set [Ku66, §40, III], then the sets E_ξ are relatively G_δ-sets in E. In this case the rank δ is determined by an increasing ω_1-sequence of G_δ-sets in E.

However, if E is a separable metric space, then no resolvable rank δ: $E \to \omega_1$ with $\delta(E)$ stationary can be determined by an increasing ω_1-sequence of F_σ-sets in E; cf. Proposition 4.1.

The rank κ: $B(\aleph_1) \to \omega_1$ with the resolution *id*: $B(\aleph_1) \to B(\aleph_1)$ show that separability of E is essential in Proposition 4.1 (and also in Lemma 2.6). Still, for resolvable ranks δ on an arbitrary Hausdorff space E, the part of the assertion concerning non-separation "from below" is valid, and, in effect, almost every layer contains a Cantor set.

7.5 Borel resolutions.

Let $\Sigma(\aleph_1)$ be the space of all functions x: $\omega_1 \to \mathbf{N}$ with countable support, equipped with the pointwise topology, i.e., $\Sigma(\aleph_1)$ is the Σ-product of $\aleph_1$ copies of natural numbers. For $x \in \Sigma(\aleph_1)$, let $\delta(x)$ be the minimal ordinal α such that x is zero on the interval $[\alpha, \omega_1)$.

In [ChGP98, Sec. 4] a Borel surjection π: $B(\aleph_1) \to \Sigma(\aleph_1)$ was constructed with all the properties required in the definition of a resolution for the rank δ except the continuity of π. The parametrization π is useful in analyzing Souslin sets in $\Sigma(\aleph_1)$. However, the rank δ: $\Sigma(\aleph_1) \to \omega_1$ has no resolution. In fact, the space of countable ordinals ω_1 with the order topology embeds in $\Sigma(\aleph_1)$ as a closed subspace so that δ is the identity on ω_1, and the identity on ω_1 is not resolvable; cf. the last statement in 7.4.

7.6 Omitting some layers by resolutions.

The resolution for the rank δ defined in Section 3(8) neglects the Borel set $\delta^{-1}(\{0\})$. Although the omission (even of non-stationary many layers) makes no difference in our arguments, one can easily fill the gaps contained in analytic sets (in non-stationary many layers) by using a surjection from the free union of the irrationals added to the domain of the resolution.

7.7 Remarks on Proposition 6.1.

The choice of the σ-discrete dense set in $B(\aleph_1)$ in Proposition 6.1 is inessential. Indeed, for any σ-discrete dense subset T of $B(\aleph_1)$ there exists a homeomorphism of $B(\aleph_1)$ onto $B(\aleph_1)$ mapping T onto Q.

Since no first category subset of $B(\aleph_1)$ contains stationary many layers B_ξ, the two conditions in Proposition 6.1 cannot be satisfied simultaneously.

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